Condition for the Existence of Complex Modes in a Trapped Bose–Einstein Condensate with a Highly Quantized Vortex

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We consider a trapped Bose–Einstein condensate (BEC) with a highly quantized vortex. For the BEC with a doubly, triply or quadruply quantized vortex, the numerical calculations have shown that the Bogoliubov–de Gennes equations, which describe the fluctuation of the condensate, have complex eigenvalues. In this paper, we obtain the analytic expression of the condition for the existence of complex modes, using the method developed by Rossignoli and Kowalski [R. Rossignoli and A. M. Kowalski, Phys. Rev. A **72**, 032101 (2005)] for the small coupling constant. To derive it, we make the two-mode approximation. With the derived analytic formula, we can identify the quantum number of the complex modes for each winding number of the vortex. Our result is consistent with those obtained by the numerical calculation in the case that the winding number is two, three or four. We prove that the complex modes always exist when the condensate has a highly quantized vortex.

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I. INTRODUCTION

The Bose–Einstein condensates (BECs) of neutral atomic gases were realized in 1995 [1, 2, 3], and several kinds of vortices inside the BECs have been observed: the singly quantized vortex [4], the vortex lattice [5, 6], and the doubly quantized vortex [7] which was created by the topological phase engineering [8]. In particular, an interesting phenomenon was observed: a doubly quantized vortex decayed into two single quantized vortices spontaneously [9].

The theoretical investigations on the instability of the highly quantized vortex of the neutral atomic BEC were made by several authors [10, 11, 12]. They solved the Bogoliubov–de Gennes (BdG) equations [13, 14, 15] numerically, and found that the equations have complex eigenvalues when a condensate has a vortex with the winding number two [10, 11], three [10] or four [12]. The complex eigenvalues cause the blowup or damping of the c-number condensate fluctuations [10, 11, 12]. This instability of the condensate, caused by the complex eigenvalues of the BdG equations, is called "dynamical instability". But it is still unknown whether a highly quantized vortex with an arbitrary high winding number always brings complex eigenvalues and which eigenvalues become complex. It should also stressed that the relation between the "dynamical instability" in theoretical concept and the observed decay of the doubly quantized vortex is not elucidated fully and still under investigation [16, 17].

As for the "complex modes", there is known another treatment, developed by Rossignoli and Kowalski [18]. We refer to it as the RK method. In this method the quantum Hamiltonian of the quadratic form of creation and annihilation operators is considered, and the complex modes appear as a result of diagonalizing it with unusual operators which are neither bosonic nor fermionic ones. Obviously the meaning of the complex modes in the RK method is different from one mentioned above [10, 11, 12]: they are quantum fluctuations in the former, while they are c-number ones in the latter. The relation between the complex modes and the complex eigenvalues of the BdG equations has not been established. Further, we point out that the interpretation of the complex eigenvalues of the BdG equations in the case that they are treated as quantum fluctuations, as well as of the complex modes of the RK method, is not simple. We recently proposed a new approach of quantum field theory, using the eigenfunctions of the BdG equations including those belonging to the complex eigenvalues and considering the free quantum Hamiltonian consistently [19], although the content of this paper is not related to the approach directly.

In this paper, we derive the conditions for the existence of the complex modes in a trapped BEC with a highly quantized vortex analytically using the RK method within the two-mode approximation and small coupling expansion. Using the condition, we can answer the question above: does any highly quantized vortex bring the complex modes? The answer is yes, and we can identify partially which mode is the complex one. The condition is also useful for checking the results of the numerical calculation.

This paper is organized as follows. In Sec. II, the model Hamiltonian describing the weak interacting neu-

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tral atoms in a harmonic trap is introduced. The c-number field whose square represents the distribution of the condensate is subject to the Gross–Pitaevskii (GP), and the quantum field is expanded in terms of a complete set. In Sec. III, we give a brief review of the RK method. In Sec. IV, first, we explain the two-mode approximation. Second, the small coupling expansion is introduced. Both of them play essential roles in our analysis. We derive the condition for the existence of the complex modes analytically within the two-mode approximation and small coupling expansion. Finally, we prove that the complex modes always exist when the condensate has a highly quantized vortex. Section V is devoted to the summary.

II. MODEL HAMILTONIAN

We start with the following Hamiltonian to describe the neutral atoms trapped by a harmonic potential of a cylindrical symmetry,

$$\hat{H} = \int d^3x \Big[\hat{\psi}^\dagger(x)(K + V(r,z) - \mu)\hat{\psi}(x) + \frac{g}{2}\hat{\psi}^{\dagger 2}(x)\hat{\psi}^2(x)\Big], \tag{1}$$

where $r = \sqrt{x^2+y^2}$, $x = (\boldsymbol{x}, t)$ and

$$K = -\frac{1}{2M}\nabla^2, \tag{2}$$

$$V(r,z) = \frac{1}{2}M(\omega_\perp^2 r^2 + \omega_z^2 z^2) \tag{3}$$

with the mass of a neutral atom M, the chemical potential μ and the coupling constant g. The bosonic field operator $\hat{\psi}(x)$ obeys the canonical commutation relations,

$$[\hat{\psi}(x),\ \hat{\psi}^\dagger(x')]|_{t=t'} = \delta(\boldsymbol{x} - \boldsymbol{x'}), \tag{4}$$

$$[\hat{\psi}(x),\ \hat{\psi}(x')]|_{t=t'} = [\hat{\psi}^\dagger(x),\ \hat{\psi}^\dagger(x')]|_{t=t'} = 0\,. \tag{5}$$

Let us divide the field operator $\hat{\psi}(x)$ into a classical field $\xi(\boldsymbol{x})$ and a quantum field $\hat{\varphi}(x)$ as

$$\hat{\psi}(x) = \xi(\boldsymbol{x}) + \hat{\varphi}(x)\,. \tag{6}$$

Here, the classical field $\xi(\boldsymbol{x})$ is the order parameter representing the condensate with a (highly) quantized vortex, which is assumed to be time-independent. Note that the function $\xi(\boldsymbol{x})$ is essentially complex due to the existence of the quantized vortex. At the tree level, $\xi(x)$ satisfies the following GP equation:

$$\left[K + V(r,z) - \mu + g|\xi(\boldsymbol{x})|^2\right]\xi(\boldsymbol{x}) = 0\,. \tag{7}$$

The condensate particle number $N_{\rm c}$ is given by

$$\int d^3x\,|\xi(\boldsymbol{x})|^2 = N_{\rm c}\,. \tag{8}$$

Substituting Eq. (6) into Eq. (1), one can rewrite the total Hamiltonian $\hat{H}$ as

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm int}\,, \tag{9}$$

where the free and interaction Hamiltonians are chosen, respectively, as

$$\begin{aligned}\hat{H}_0 = \int d^3x \Big[&\hat{\varphi}^\dagger(x)\left\{K + V(r,z) - \mu\right\}\hat{\varphi}(x) \\ &+ \frac{g}{2}\big\{4|\xi(\boldsymbol{x})|^2\hat{\varphi}^\dagger(x)\hat{\varphi}(x) \\ &+ \xi^{*2}(\boldsymbol{x})\hat{\varphi}^2(x) + \xi^2(\boldsymbol{x})\hat{\varphi}^{\dagger 2}(x)\big\}\Big],\end{aligned} \tag{10}$$

$$\begin{aligned}\hat{H}_{\rm int} = g\int d^3x \Big[&\xi(\boldsymbol{x})\hat{\varphi}^{\dagger 2}(x)\hat{\varphi}(x) + \xi^*(\boldsymbol{x})\hat{\varphi}^\dagger(x)\hat{\varphi}^2(x) \\ &+ \frac{1}{2}\hat{\varphi}^{\dagger 2}(x)\hat{\varphi}^2(x)\Big]\,.\end{aligned} \tag{11}$$

Assuming that a vortex is created along the z-axis and taking account of an axial symmetry along the vortex line, we introduce a real function $f(r,z)$, defined as

$$\xi(\boldsymbol{x}) = \sqrt{\frac{N_{\rm c}}{2\pi}}e^{i\kappa\theta}f(r,z)\,, \tag{12}$$

where the integer $\kappa \geq 1$ is a winding number of the vortex. The function $f(r,z)$ is normalized as

$$\int r dr\, dz\, f^2(r,z) = 1\,. \tag{13}$$

Substituting Eq. (12) into Eq. (7), one obtains

$$\left[K_\kappa + V(r,z) - \mu + \frac{gN_{\rm c}}{2\pi}f^2(r,z)\right]f(r,z) = 0\,, \tag{14}$$

where

$$K_\kappa = -\frac{1}{2M}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\kappa^2}{r^2} + \frac{\partial^2}{\partial z^2}\right)\,. \tag{15}$$

Using the bosonic creation and annihilation operators $\hat{a}^\dagger_{n,\ell,m}$ and $\hat{a}_{n,\ell,m}$, we expand the bosonic field $\hat{\varphi}(x)$,

$$\hat{\varphi}(x) = \sum_{n,m=0}^{\infty}\sum_{\ell=-\infty}^{\infty}\hat{a}_{n,\ell,m}e^{i(\ell+\kappa)\theta}u_{n,\ell,m}(r,z)\,, \tag{16}$$

where n $(=0,1,2,\cdots)$ is the principle quantum number (representing the radial nodes), ℓ $(=0,\pm1,\pm2,\cdots)$ is the magnetic quantum number, m $(=0,1,2,\cdots)$ is the quantum number along the z axis. The function $u_{n,\ell,m}(r,z)$ is a solution of the eigenequation with an eigenvalue $\epsilon_{n,\ell,m}$,

$$\left[K_{\kappa+\ell} + V(r,z)\right]u_{n,\ell,m}(r,z) = \varepsilon_{n,\ell,m}u_{n,\ell,m}(r,z)\,, \tag{17}$$

and satisfies the following orthogonal and completeness conditions:

$$2\pi\int r dr\, dz\, u_{n,\ell,m}(r,z)u_{n',\ell,m'}(r,z) = \delta_{n,n'}\delta_{m,m'}\,, \tag{18}$$

$$\sum_{n,m=0}^{\infty}u_{n,\ell,m}(r,z)u_{n,\ell,m}(r',z') = \frac{1}{2\pi r}\delta(r-r')\delta(z-z')\,. \tag{19}$$

The eigenfunctions in Eq. (17) are expressed with the Laguerre polynomials $L_n^k(x)$ and the Hermite polynomials $H_m(x)$ [20] as

$$u_{n,\ell,m}(r,z) = C_{n,\ell,m} e^{-\frac{1}{2}\alpha_\perp^2 r^2} (\alpha_\perp r)^{|\ell+\kappa|} L_n^{|\ell+\kappa|}(\alpha_\perp^2 r^2) \times e^{-\frac{1}{2}\alpha_z^2 z^2} H_m(\alpha_z z)\,, \tag{20}$$

where

$$C_{n,\ell,m} = \sqrt{\frac{\alpha_\perp^2 \alpha_z}{\pi^{3/2}}} \cdot \sqrt{\frac{n!}{2^m\ m!\ (|\ell+\kappa|+n)!}}\,, \tag{21}$$

$\alpha_\perp = \sqrt{M\omega_\perp}$ and $\alpha_z = \sqrt{M\omega_z}$. The eigenvalues are given as

$$\varepsilon_{n,\ell,m} = \omega_\perp\left(2n + |\ell+\kappa| + 1\right) + \omega_z\left(m + \frac{1}{2}\right)\,. \tag{22}$$

The free Hamiltonian is a sum of the sectors $\hat{H}^{|\ell|}$, labeled by the absolute value of the magnetic quantum number $j = |\ell|$:

$$\hat{H}_0 = \sum_{j=0}^{\infty} \hat{H}^j = \hat{H}^0 + \sum_{j'=1}^{\infty} \hat{H}^{j'}\,. \tag{23}$$

Hereafter we use the symbols j as $j = 0, 1, 2, \cdots$, and j' as $j' = 1', 2', \cdots$, to make it explicit whether the special sector with $j = 0$ is included or not in the arguments.

The sectors $\hat{H}^0$ and $\hat{H}^{j'}$ $(j' \geq 1)$ are explicitly given as

$$\hat{H}^0 = \sum_{n,n',m,m'=0}^{\infty} \Big[\hat{a}^\dagger_{n,0,m} \hat{a}_{n',0,m'} A(n,0,m;n',0,m') + \frac{1}{2}\left\{\hat{a}_{n,0,m}\hat{a}_{n',0,m'} B(n,0,m;n',0,m') + \text{h.c.}\right\}\Big]\,, \tag{24}$$

$$\begin{aligned}\hat{H}^{j'} = \sum_{n,n',m,m'=0}^{\infty} \Big[&\hat{a}^\dagger_{n,j',m} \hat{a}_{n',j',m'} A(n,j',m;n',j',m') \\ &+ \hat{a}^\dagger_{n,-j',m} \hat{a}_{n',-j',m'} A(n,-j',m;n',-j',m') \\ &+ \frac{1}{2}\{\hat{a}_{n,j',m}\hat{a}_{n',-j',m'} B(n,j',m;n',-j',m') + \text{h.c.}\} \\ &+ \frac{1}{2}\{\hat{a}_{n,-j',m}\hat{a}_{n',j',m'} B(n,-j',m;n',j',m') \\ &+ \text{h.c.}\}\Big]\,, \end{aligned} \tag{25}$$

with

$$A(n,\ell,m;n',\ell',m') = \{(\varepsilon_{n,\ell,m} - \mu)\delta_{n,n'}\delta_{m,m'} + 4K(n,\ell,m;n',\ell',m')\}\delta_{\ell,\ell'}\,, \tag{26}$$

$$B(n,\ell,m;n',\ell',m') = 2K(n,\ell,m;n',\ell',m')\delta_{\ell,-\ell'}\,, \tag{27}$$

where $K(n,\ell,m;n',\ell',m')$ is defined as

$$K(n,\ell,m;n',\ell',m') = \frac{gN_\mathrm{c}}{2}\int r dr\, dz\, f^2(r,z) u_{n,\ell,m}(r,z) u_{n',\ell',m'}(r,z)\,. \tag{28}$$

III. ROSSIGNOLI–KOWALSKI METHOD

In this section, we briefly review the RK method. In this method the "complex modes" appear in the diagonalized Hamiltonian with unusual operators which are neither bosonic nor fermionic ones [18].

We begin with the general Hamiltonian for the BEC of the quadratic form of creation and annihilation operators $\hat{a}_N^\dagger$ and $\hat{a}_N$ as

$$\begin{aligned}\hat{H}_\text{quad} &= \sum_{N,N'}\left\{A_{NN'}\left(\hat{a}^\dagger_N \hat{a}_{N'} + \frac{1}{2}\delta_{N,N'}\right) + \frac{1}{2}(B_{NN'}\hat{a}^\dagger_N\hat{a}^\dagger_{N'} + \text{h.c.})\right\} \\ &= \frac{1}{2} Z^\dagger \mathcal{H} Z\,, \end{aligned} \tag{29}$$

where

$$\mathcal{H} = \begin{pmatrix} A & B \\ B^* & A^t \end{pmatrix}\,, \tag{30}$$

$$Z = \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}\,. \tag{31}$$

Here, the matrices A and B are hermite and symmetric, respectively. The symbols $\hat{a}$ and $\hat{a}^\dagger$ represent the arrays of components $\hat{a}_N$ and $\hat{a}_N^\dagger$:

$$\hat{a} = \begin{pmatrix} \hat{a}_{N=1} \\ \hat{a}_{N=2} \\ \vdots \end{pmatrix}\,, \tag{32}$$

$$\hat{a}^\dagger = \begin{pmatrix} \hat{a}^\dagger_{N=1} \\ \hat{a}^\dagger_{N=2} \\ \vdots \end{pmatrix}\,, \tag{33}$$

where N stands for the combination of the quantum numbers n, ℓ and m, and we treat N as a single integer for simplicity. The bosonic commutation relations of the original creation and annihilation operators $\hat{a}_N$ and $\hat{a}_N^\dagger$ are written as

$$ZZ^\dagger - (Z^{\dagger t} Z^t)^t = \mathcal{M}\,, \tag{34}$$

where

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{35}$$

Now, we consider a general linear canonical transformation of the $\hat{a}$ and $\hat{a}^\dagger$

$$Z = \mathcal{W} Z'\,, \tag{36}$$

$$Z' = \begin{pmatrix} \hat{a}' \\ \bar{a}' \end{pmatrix}\,, \tag{37}$$

where

$$\hat{a}' = \begin{pmatrix} \hat{a}'_{N=1} \\ \hat{a}'_{N=2} \\ \vdots \end{pmatrix}, \tag{38}$$

$$\bar{a}' = \begin{pmatrix} \bar{a}'_{N=1} \\ \bar{a}'_{N=2} \\ \vdots \end{pmatrix}. \tag{39}$$

We note that $\bar{a}'_N$ is not necessarily the hermitian conjugate of $\hat{a}'_N$, although $\hat{a}'_N$ and $\bar{a}'_N$ are required to satisfy the bosonic commutation relations as

$$Z'\bar{Z}' - (\bar{Z}'^t Z'^t)^t = \mathcal{M}, \tag{40}$$

where

$$\bar{Z}' = \begin{pmatrix} \bar{a}' & \hat{a}' \end{pmatrix} = Z'^t \mathcal{T}, \tag{41}$$

$$\mathcal{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{42}$$

As $Z'^\dagger = \bar{Z}'\bar{\mathcal{W}}$, the matrix $\mathcal{W}$ should satisfy

$$\mathcal{W}\mathcal{M}\bar{\mathcal{W}} = \mathcal{M}, \tag{43}$$

where

$$\bar{\mathcal{W}} = \mathcal{T}\mathcal{W}^t\mathcal{T}. \tag{44}$$

It is important to notice that in the case where only the real modes appear, the relation $\bar{\mathcal{W}} = \mathcal{W}^\dagger$ holds, and then $\bar{a}'_N = \hat{a}'^\dagger_N$.

The quadratic Hamiltonian $\hat{H}_{\text{quad}}$ is rewritten as

$$\hat{H}_{\text{quad}} = \frac{1}{2}\bar{Z}'\mathcal{H}'Z', \tag{45}$$

$$\mathcal{H}' = \bar{\mathcal{W}}\mathcal{H}\mathcal{W}. \tag{46}$$

Finding a representation in which $\tilde{\mathcal{H}}'$ is diagonal corresponds to solving an eigenvalue equation with the "metric" $\mathcal{M}$, i.e.,

$$\mathcal{H}\mathcal{W} = \mathcal{M}\mathcal{W}\mathcal{M}\mathcal{H}'. \tag{47}$$

We regard this equation as an eigenvalue equation for a non-hermitian matrix $\tilde{\mathcal{H}}$:

$$\tilde{\mathcal{H}}\mathcal{W} = \mathcal{W}\tilde{\mathcal{H}}', \tag{48}$$

$$\tilde{\mathcal{H}} = \mathcal{M}\mathcal{H} = \begin{pmatrix} A & B \\ -B^* & -A^t \end{pmatrix}, \tag{49}$$

with the diagonal matrix $\tilde{\mathcal{H}}'$ whose diagonal components are eigenvalues of $\tilde{\mathcal{H}}$, denoted by χ_N. Then the quadratic Hamiltonian $\hat{H}_{\text{quadratic}}$ can be expressed as

$$\hat{H}_{\text{quad}} = \sum_N \chi_N \left(\bar{a}'_N \hat{a}'_N + \frac{1}{2} \right). \tag{50}$$

IV. CONDITION FOR THE EXISTENCE OF THE COMPLEX MODES

Applying the RK method [18] reviewed in the previous section to our Hamiltonian in Eqs. (23)–(28), we now solve the eigenvalue equation (48) under the circumstance to obtain the mode-decoupled form of our Hamiltonian. But the equation is too difficult to solve analytically in general.

In this section, we develop the following two approximate schemes in Eq. (48). First, we make a two-mode approximation on the Hamiltonian (23). Second, we introduce the small coupling expansion. Both of them are helpful for the analytical study in spite of the nonlinearity of the basic equation for the condensate (7). And finally, we analytically derive the condition for the existence of the complex modes.

A. Two-Mode Approximation

In this subsection, we explain the two-mode approximation which is used to obtain the analytic expression of the condition for the existence of the complex modes.

First, we consider the $j = 0$ part of the free Hamiltonian (24). Within the two-mode approximation, $\hat{H}^0$ is written in the matrix representation as

$$\hat{H}^0 = \frac{1}{2} Z_0^\dagger \mathcal{H}_0 Z_0, \tag{51}$$

where

$$Z_0 = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix}, \tag{52}$$

$$Z_0^\dagger = \begin{pmatrix} \hat{a}_1^\dagger & \hat{a}_2^\dagger & \hat{a}_1 & \hat{a}_2 \end{pmatrix}, \tag{53}$$

$$\mathcal{H}_0 = \begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix}, \tag{54}$$

$$A_0 = \begin{pmatrix} \varepsilon_1 - \mu + 4K(1,1) & 4K(1,2) \\ 4K(2,1) & \varepsilon_2 - \mu + 4K(2,2) \end{pmatrix}, \tag{55}$$

$$B_0 = \begin{pmatrix} 2K(1,1) & 2K(1,2) \\ 2K(2,1) & 2K(2,2) \end{pmatrix}. \tag{56}$$

Here, the mode 1 represents one labeled by the quantum numbers $(n, \ell, m) = (n_1, 0, m_1)$ and the mode 2 does $(n, \ell, m) = (n_2, 0, m_2)$. We also define the tilde matrix

$$\tilde{\mathcal{H}}_0 = \begin{pmatrix} A_0 & B_0 \\ -B_0 & -A_0 \end{pmatrix}. \tag{57}$$

Next, we consider the $j \neq 0$ part of the free Hamiltonian (25). In the matrix representation, $\hat{H}^{j'}$ under the two-mode approximation is written as

$$\hat{H}^{j'} = \frac{1}{2} Z_{j'}^\dagger \mathcal{H}_{j'} Z_{j'}, \tag{58}$$

where

$$Z_{j'} = \begin{pmatrix} \hat{a}_{1'} \\ \hat{a}_{2'} \\ \hat{a}^{\dagger}_{1'} \\ \hat{a}^{\dagger}_{2'} \end{pmatrix}, \tag{59}$$

$$Z^{\dagger}_{j'} = \begin{pmatrix} \hat{a}^{\dagger}_{1'} & \hat{a}^{\dagger}_{2'} & \hat{a}_{1'} & \hat{a}_{2'} \end{pmatrix}, \tag{60}$$

$$\mathcal{H}_{j'} = \begin{pmatrix} A_{j'} & B_{j'} \\ B_{j'} & A_{j'} \end{pmatrix}, \tag{61}$$

$$A_{j'} = \begin{pmatrix} \varepsilon_{1'} - \mu + 4K(1',1') & 0 \\ 0 & \varepsilon_{2'} - \mu + 4K(2',2') \end{pmatrix}, \tag{62}$$

$$B_{j'} = \begin{pmatrix} 0 & 2K(1',2') \\ 2K(2',1') & 0 \end{pmatrix}. \tag{63}$$

Here, the quantum numbers (n, ℓ, m) of the mode 1' and the mode 2' are (n'_1, j', m'_1) and $(n'_2, -j', m'_2)$, respectively. The tilde matrix is defined by

$$\tilde{\mathcal{H}}_{j'} = \begin{pmatrix} A_{j'} & B_{j'} \\ -B_{j'} & -A_{j'} \end{pmatrix}. \tag{64}$$

B. Small coupling expansion

In this subsection, we define the dimensionless parameter λ as

$$\lambda = d_{\mathrm{ho}}(gN_{\mathrm{c}}), \tag{65}$$

and make an assumption of

$$|\lambda| \ll 1, \tag{66}$$

where $d_{\mathrm{ho}} = (\alpha_\perp^2 \alpha_z)^{1/3} M$ is the constant determined by the typical scales of the system, and expand the function $f(r,z)$ (see Eq. (14)) and chemical potential μ with respect to this small parameter λ. We refer to this expansion as the small coupling expansion. Here we perform this expansion up to the first order in λ. Note that this expansion is applicable not only for repulsive interaction ($\lambda > 0$), but also for attractive one ($\lambda < 0$). Let us introduce the notations of quantities of the zeroth order in λ, namely, a real function $f_0(r,z)$ and a chemical potential μ_0, which satisfy

$$[K_\kappa + V(r,z)] f_0(r,z) = \mu_0 f_0(r,z). \tag{67}$$

The function $f_0(r,z)$ and the quantity μ_0 correspond to the eigenfunction $u_{0,0,0}(r,z)$ and the eigenvalue $\varepsilon_{0,0,0}$, respectively, i.e.,

$$\begin{aligned} f_0(r,z) &= \sqrt{2\pi}\, u_{0,0,0}(r,z) \\ &= \sqrt{2\pi}\, C_{0,0,0} e^{-\frac{1}{2}\alpha_\perp^2 r^2} (\alpha_\perp r)^\kappa e^{-\frac{1}{2}\alpha_z^2 z^2}, \end{aligned} \tag{68}$$

$$\mu_0 = \varepsilon_{0,0,0} = \omega_\perp(\kappa+1) + \frac{\omega_z}{2}. \tag{69}$$

Note that the function $f_0(r,z)$ is normalized as

$$\int r dr\, dz\, f_0^2(r,z) = 1. \tag{70}$$

Using the function $f_0(r,z)$ we can expand the real function $f(r,z)$ up to the first order in λ:

$$f(r,z) \simeq G\left\{ f_0(r,z) + \lambda f_1(r,z) \right\}, \tag{71}$$

where $f_1(r,z)$ is a real function and normalized,

$$\int r dr\, dz\, f_1^2(r,z) = 1, \tag{72}$$

and the parameter G is determined by the normalization condition (13) as

$$G = (1 + 2\lambda p_1)^{-\frac{1}{2}}, \tag{73}$$

where the coefficient p_1 is defined as

$$p_1 = \int r dr\, dz\, f_0(r,z) f_1(r,z), \tag{74}$$

which leads to

$$f(r,z) \simeq f_0(r,z) + \lambda\{ f_1(r,z) - p_1 f_0(r,z) \}. \tag{75}$$

We also need to expand the chemical potential μ up to the first order in λ:

$$\mu \simeq \mu_0 + \lambda \mu_1. \tag{76}$$

We determine μ_1 by substituting Eq. (75) and Eq. (76) into the GP equation (14),

$$\{K_\kappa + V(r,z) - \mu_0\} f_1(r,z) - \mu_1 f_0(r,z) + \frac{1}{2\pi d_{\mathrm{ho}}} f_0^3(r,z) = 0. \tag{77}$$

Here we expand the function $f_1(r,z)$ with the eigenfunctions $u_{n,0,m}(r,z)$,

$$f_1(r,z) = \sum_{n,m=0}^{\infty} s_{n,m} u_{n,0,m}(r,z), \tag{78}$$

where the expansion coefficients $s_{n,m}$ are written as

$$s_{n,m} = 2\pi \int r dr\, dz\, u_{n,0,m}(r,z) f_1(r,z). \tag{79}$$

It is notable to see that

$$s_{0,0} = \sqrt{2\pi}\, p_1. \tag{80}$$

Substitute Eq. (78) to Eq. (77), one obtains

$$\begin{aligned} &\sum_{n,m=0}^{\infty} (\varepsilon_{n,0,m} - \mu_0) s_{n,m} u_{n,0,m}(r,z) - \mu_1 f_0(r,z) \\ &+ \frac{1}{2\pi d_{\mathrm{ho}}} f_0^3(r,z) = 0. \end{aligned} \tag{81}$$

Multiplying the function $f_0(r,z)$ to Eq. (81) and integrating over the space coordinates, we obtain

$$\mu_1 = \frac{1}{2\pi d_{\rm ho}} \int r dr\, dz\, f_0^4(r,z) = \frac{2\pi}{d_{\rm ho}} C \frac{(2\kappa)!}{(\kappa!)^2} \tag{82}$$

where C is defined as

$$C = \sqrt{\frac{1}{2\pi^5}} \cdot \frac{\alpha_\perp^2 \alpha_z}{2^{2\kappa+2}}\,. \tag{83}$$

Thus $f_1(r,z)$ and μ_1 are determined.

C. The condition for the existence of the complex modes

In this subsection, we calculate the eigenvalues of the matrices $\tilde{\mathcal{H}}_0$ in Eq. (57) and $\tilde{\mathcal{H}}_{j'}$ in Eq. (64) under the small coupling expansion. To simplify the expression, we parameterize the elements of the matrices A_j and B_j ($j = 0, j'$) as

$$A_{j,11} = a_j + r_{j,a}\,, \tag{84}$$
$$A_{j,22} = a_j - r_{j,a}\,, \tag{85}$$
$$A_{j,12} = A_{j,21} = \Delta_{j,a}\,, \tag{86}$$
$$B_{j,11} = b_j + r_{j,b}\,, \tag{87}$$
$$B_{j,22} = b_j - r_{j,b}\,, \tag{88}$$
$$B_{j,12} = B_{j,21} = \Delta_{j,b}\,. \tag{89}$$

The eigenvalues χ_j of the matrix $\tilde{\mathcal{H}}_j$ are characterized by the equation

$$\det[\tilde{\mathcal{H}}_j - \chi_j] = 0\,. \tag{90}$$

Then we obtain

$$\chi_j^2 = (a_j^2 + r_{j,a}^2 + \Delta_{j,a}^2) - (b_j^2 + r_{j,b}^2 + \Delta_{j,b}^2) \pm 2\sqrt{I}\,, \tag{91}$$

$$I = (a_j r_{j,a} - b_j r_{j,b})^2 + (a_j \Delta_{j,a} - b_j \Delta_{j,b})^2 - (r_{j,a}\Delta_{j,b} - r_{j,b}\Delta_{j,a})^2\,. \tag{92}$$

We discuss the two cases, $j = 0$ and $j \neq 0$, separately.

1. $j = 0$ case

In the case of $j = 0$, the elements of the matrix A_0 are written as

$$a_0 = \frac{1}{2}\{2\omega_\perp(n_1+n_2) + \omega_z(m_1+m_2)\} + O(\lambda) \tag{93}$$
$$= \bar{a}_0 + O(\lambda)\,, \tag{94}$$
$$r_{0,a} = \frac{1}{2}\{2\omega_\perp(n_1-n_2) + \omega_z(m_1-m_2)\} + O(\lambda) \tag{95}$$
$$= \bar{r}_{0,a} + O(\lambda)\,. \tag{96}$$

It is easy to see that if $|\lambda| \ll 1$, then

$$\chi_0^2 = (\bar{a}_0 \pm \bar{r}_{0,a})^2 + O(\lambda) > 0\,. \tag{97}$$

¿From this relation, we conclude that no complex mode appears for $j = 0$ and sufficiently small λ.

2. $j \neq 0$ case

In the case of $j \neq 0$, it is straightforward to see

$$\chi_{j'}^2 = \left(r_{j',a} \pm \sqrt{a_{j'}^2 - \Delta_{j',b}^2}\right)^2 \tag{98}$$

for $j' \geq 1$. So the sign of $a_{j'}^2 - \Delta_{j',b}^2$ determines whether the eigenvalues are real or not. Note that it has the term of $\varepsilon_{1'} + \varepsilon_{2'}$, and when we search complex eigenvalues, it is natural to choose the mode 1' as $(0, j', 0)$ and the mode 2' as $(0, -j', 0)$, because then $a_{j'}^2$ reaches its minimum.

In the case of $j' > \kappa$, $\varepsilon_{1'}$ and $\varepsilon_{2'}$ are written as

$$\varepsilon_{1'} = \omega_\perp(j'+\kappa+1) + \frac{\omega_z}{2}, \tag{99}$$
$$\varepsilon_{2'} = \omega_\perp(j'-\kappa+1) + \frac{\omega_z}{2}\,. \tag{100}$$

Since the matrix element $a_{j'}$ is written as

$$a_{j'} = \omega_\perp(j'-\kappa) - \lambda\mu_1 + 2K(1',1') + 2K(2',2')\,, \tag{101}$$

the term $a_{j'}^2 - \Delta_{j',b}^2$ is evaluated as

$$a_{j'}^2 - \Delta_{j',b}^2 = \omega_\perp^2 (j'-\kappa)^2 + O(\lambda) > 0\,. \tag{102}$$

So we can see that in the case of $j' > \kappa$ no complex mode exists.

In the case of $0 < j' \leq \kappa$, $\varepsilon_{1'}$ and $\varepsilon_{2'}$ are

$$\varepsilon_{1'} = \omega_\perp(j'+\kappa+1) + \frac{\omega_z}{2}\,, \tag{103}$$
$$\varepsilon_{2'} = \omega_\perp(-j'+\kappa+1) + \frac{\omega_z}{2}\,. \tag{104}$$

Then the matrix element $a_{j'}$ becomes

$$a_{j'} = -\lambda\mu_1 + 2K(1',1') + 2K(2',2') \tag{105}$$

where $K(1',1')$ and $K(2',2')$ are written as

$$K(1',1') = \frac{2\pi}{d_{\rm ho}} \lambda \frac{C}{2} \frac{(2\kappa+j')!}{2^{j'}(\kappa+j')!\kappa!}\,, \tag{106}$$
$$K(2',2') = \frac{2\pi}{d_{\rm ho}} \lambda \frac{C}{2} \frac{(2\kappa-j')!}{2^{-j'}(\kappa-j')!\kappa!}\,. \tag{107}$$

The parameter $\Delta_{j',b}$ is given as

$$\Delta_{j',b} = \frac{2\pi}{d_{\rm ho}} \lambda C \frac{(2\kappa)!}{\kappa! \sqrt{(\kappa+j')!(\kappa-j')!}}\,. \tag{108}$$

It is important to see that $a_{j'}$ and $\Delta_{j',b}$ have no term of the zeroth order in the parameter λ (cf. Eqs. (93) and (101)). Thus we must evaluate all terms in $a_{j'}^2 - \Delta_{j',b}^2$ up to the second order in λ. Then, $a_{j'}^2 - \Delta_{j',b}^2$ is evaluated as

$$a_{j'}^2 - \Delta_{j',b}^2 = \lambda^2 \left\{\mu_1 - \frac{2\pi}{d_{\rm ho}} C(S+T)\right\} \times \left\{\mu_1 - \frac{2\pi}{d_{\rm ho}} C(S-T)\right\}\,, \tag{109}$$

$\kappa\backslash j$	1	2	3	4	5	$\cdots$
1						
2		$\times$				
3		$\times$	$\times$			
4		$\times$	$\times$			
5		$\times$	$\times$	$\times$		
6		$\times$	$\times$	$\times$		
$\vdots$						

TABLE I: The existence of the complex modes. The condition (112) are examined for various values of (κ, j). The marks are the positions where the complex modes arise. This result is valid for the repulsive interaction ($\lambda > 0$), as well as the attractive one ($\lambda < 0$), under the situation of $|\lambda| \ll 1$.

where S and T are defined as

$$S = \frac{(2\kappa + j')!}{2^{j'}(\kappa + j')!\kappa!} + \frac{(2\kappa - j')!}{2^{-j'}(\kappa - j')!\kappa!}\,, \tag{110}$$

$$T = \frac{(2\kappa)!}{\kappa!\sqrt{(\kappa + j')!(\kappa - j')!}}\,. \tag{111}$$

Therefore, the condition for the existence of the complex modes, $a_{j'}^2 - \Delta_{j',b}^2 < 0$, now amounts to

$$S - T < \frac{(2\kappa)!}{(\kappa!)^2} < S + T\,. \tag{112}$$

We apply the condition (112) to each winding number and find the appearance of the complex modes (Table I). In the case of the winding number $\kappa = 1$, no complex mode exists. On the other hand, in the case of the winding number $\kappa \geq 2$ the complex modes always exist. Particularly, we find that in the case of the winding number $\kappa = 2$ the complex modes appear as a pair, while in case of $\kappa = 3$ the complex modes arise as two pairs. So our results in the case of the winding number $\kappa = 2$ and 3 are consistent with those by Pu *et al.* [10] and Möttönen *et al.* [11]. In the case $\kappa = 4$, our result is consistent with that by Kawaguchi and Ohmi [12]. Remark that Pu *et al.* [10] performs their analysis also for the attractive interaction ($\lambda < 0$), while the other authors do only for the repulsive interaction ($\lambda > 0$) [11, 12]. Our result is also consistent with that by Pu *et al.* [10] for $\lambda < 0$.

The condition (112) can be applicable only in the small coupling constant region. So the existence of new complex modes which may arise in the large coupling constant region, such as one can find them in Fig. 2 in Ref. [12], is outside the scope of this paper.

D. Proof of the existence of the complex modes for $\kappa \geq 2$

One of the most important consequence of the analytic result (112) is the following statement: *The $j' = 2$ modes are always complex ones for $\kappa \geq 2$.* The proof is given below.

For $j' = 2$, the inequality (112) is reduced to the following coupled inequalities

$$1 < \frac{(2\kappa+2)(2\kappa+1)}{4(\kappa+2)(\kappa+1)} + \frac{4\kappa(\kappa-1)}{(2\kappa)(2\kappa-1)} + \sqrt{\frac{\kappa(\kappa-1)}{(\kappa+2)(\kappa+1)}}\,, \tag{113}$$

$$\frac{(2\kappa+2)(2\kappa+1)}{4(\kappa+2)(\kappa+1)} + \frac{4\kappa(\kappa-1)}{(2\kappa)(2\kappa-1)} - \sqrt{\frac{\kappa(\kappa-1)}{(\kappa+2)(\kappa+1)}} < 1\,. \tag{114}$$

First, let us prove the inequality (113). Remark that the inequality

$$\frac{(2\kappa+2)(2\kappa+1)}{4(\kappa+2)(\kappa+1)} + \frac{4\kappa(\kappa-1)}{(2\kappa)(2\kappa-1)} - 1 > 0 \tag{115}$$

is reduced to

$$\kappa(\kappa+1)(4\kappa^2 - 2\kappa - 5) > 0\,. \tag{116}$$

The largest root for the left-hand side is $\kappa = \kappa_1 \simeq 1.396$. Combining this result with the fact $\sqrt{\{\kappa(\kappa-1)\}/\{(\kappa+2)(\kappa+1)\}} > 0$ for $\kappa > \kappa_1$, we can find the inequality (113) is satisfied for $\kappa > \kappa_1$.

Next, let us prove the inequality (114). The inequality (114) is equivalent to

$$\kappa^2(\kappa+1)(8\kappa^3 + 52\kappa^2 - 53\kappa - 25) > 0\,. \tag{117}$$

The largest root for the left-hand side is $\kappa = \kappa_2 \simeq 1.199$. ¿From the same discussion on the inequality (113), we find that the inequality (114) is satisfied by $\kappa > \kappa_2$.

¿From our discussion above, the roots κ_1 and κ_2 satisfy the inequality $1 < \kappa_1, \kappa_2 < 2$. This means that if $\kappa \geq 2$, κ satisfies the inequalities (113) and (114) simultaneously, then the statement is proven.

V. SUMMARY

We derived the analytic expression of the condition for the existence of the complex modes when the condensate has a highly quantized vortex, using the method developed by Rossignoli and Kowalski [18] for the small coupling constant, under the two-mode approximation. Our results agree with those by Pu *et al.* ($\kappa = 2, 3$) [10], Möttönen *et al.* ($\kappa = 2$) [11] and Kawaguchi and Ohmi ($\kappa = 4$) [12]. It is emphasized that the formula derived here is applicable for an arbitrary winding number, and one can check whether complex modes exist or not for an arbitrary highly quantized vortex in the condensate. We have also proven that in the case of $\kappa \geq 2$, the complex modes always exist. This means that the arbitrarily highly quantized vortex may be dynamically unstable. Moreover, it is shown that imaginary parts of

the complex eigenvalues rise linearly against the coupling constant.

The validity of the two-mode approximation is certainly not clear. But the numerical calculation for the case of $\kappa = 2$ and $\lambda > 0$ revealed that the behavior of the imaginary part in the small coupling constant region is not significantly affected by the number of the modes included to the calculation [10]. This fact suggests that the two-mode approximation does not modify behaviors of complex modes in the small coupling constant region essentially and that the analysis in this paper is reliable.

Finally, we comment that the physical interpretation of the complex modes is very difficult and is not settled yet. Pu *et al.* employs the classical interpretation on the complex modes, with the idea that the complex modes cause the dynamical instability, that is, the decay of the initial configuration of the condensate [10]. The argument does not touch the aspect of quantum theory. We recently proposed a quantum field theoretical treatment associated with the complex modes and sought a new interpretation [19].

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